

Closed-form Approximations for Spread Options in Lévy Markets

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Abstract

We provide new closed-form approximations for the pricing of spread options in three specific instances of exponential Lévy markets, i.e., when log-returns are modeled as Brownian motions (Black-Scholes model), Variance Gamma processes (VG model) or Normal Inverse Gaussian processes (NIG model). For the specific case of exchange options (spread options with zero strike), we generalize the well-known Margrabe formula (1978) that is valid in a Black-Scholes model to the VG model under a homogeneity assumption.

Keywords: Stochastic clock, Lévy markets, Conditional expectation, Gaussian quadrature, Margrabe's formula

1 Introduction

A spread option is an exotic derivative with a payoff that is based on the price difference (i.e., the spread) between two or more underlying assets. In this paper we consider European spread options, which means that the buyer has the right to receive the spread by paying the strike price on the exercise date. Spread option contracts are ubiquitous in financial markets where they can serve as a speculative device or risk management tool. Positions in these products can be purchased on some large exchanges but are primarily traded over-the-counter.

For instance, a popular product in the U.S. fixed income market is the Note Against Bond (NOB) spread in which a yield curve spread is created between the 30-year bond futures contract

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(long position) and the 10-year U.S. Treasury note futures (short position), and which can be useful for both speculation and hedging purposes. In commodity markets, we can distinguish between calendar spreads, location spreads, quality spreads and processing spreads. The first three relate to differences in the prices of the same commodity at two different time stamps, different locations, and different grades, respectively, whereas the latter relates to differences in the prices of at least two different commodities (which are inputs and outputs of production processes). A popular processing spread option is the crack spread, which is based on the differential between the price of crude oil and refined petroleum products. Crack spreads are widely used by refineries to hedge against price fluctuations, mitigate risk or secure a profit margin on the production output. Moreover, they allow refineries to manage capacity and production since the spread approximates the average gross processing margin, and thus the market efficiency. Obviously, crack spreads also allow speculators to bet on the evolution of the price differential. For more details on spread options contracts in financial markets, we refer to Carmona and Durrleman (2003b).

In addition to their widespread use in financial markets, spread options also appear in real option valuation (i.e., financial option valuation techniques are used for capital budgeting decision) and in credit risk management. In real option valuation, a well-known example is the spark spread. This product can be used for electricity plant valuation as it is considered a proxy for the cost of converting a specific fuel into electricity at a specific facility; see also e.g., Vollert (2001) for more information. In a credit risk management context, Eberlein and Madan (2012) explain that if firms have access to unbounded liabilities and are granted limited liability, they receive a free option to put losses back to the taxpayers. They call this option the taxpayer option and show its valuation amounts to pricing a spread option.

Spread option contracts may involve more than two assets, but in this paper we solely focus on the two-dimensional case. The two-dimensionality involved makes spread options harder to value than plain vanilla calls and puts, and analytical solutions for their pricing are not available for standard market models. Consequently, several approximations and numerical methods have been proposed in the literature (an overview is provided in Section 2). The majority of these methods, however, are only valid when asset prices are assumed to evolve according to a (geometric) Brownian motion (i.e., under the seminal Black-Scholes model). However, although this market setting is a corner stone in the finance literature, its deficiencies are also well documented. Specifically, since asset price volatility tends to be time-varying and to exhibit clustering effects, asset returns display fatter tails than can be modeled by a normal distribution; see e.g., Mandelbrot (1963). Moreover, there is also the frequent observation that negative returns have heavier tails than positive ones. Even though evidence of skewness in real-world asset returns is somewhat equivocal (Eberlein & Keller, 1995, and Küchler, Neumann, Sørensen, & Steller, 1999), for option pricing, the relevant returns are typically significantly negatively skewed Carr, Geman, Madan, and Yor (2002). Finally, yet another problem with the Black-Scholes setting is that it implies asset prices to evolve without jumps. In this regard, Lévy processes, which can be seen as mere generalizations of Brownian motions, provide more flexibility to cope with the aforementioned stylized features of asset returns. In what follows we derive spread option pricing formulas for three specific Lévy models, i.e., when the log-returns are modeled as Brownian motions (Black-Scholes model), Variance Gamma processes (VG model) or Normal Inverse Gaussian processes (NIG model). For a pedagogic review of these processes see for instance Deng and Yao (2017).

Spread option pricing can also be achieved using Monte Carlo simulations. Whilst this

technique combines flexibility with accuracy (i.e., asymptotically the simulated value converges to the true value) it also has some drawbacks. The budget for carrying out simulations might be significant, a situation that for instance appears when assessing the risk of a portfolio of financial instruments over a given time horizon. In this instance, analytical approximation methods are certainly useful; see also Deelstra, Rayée, Vanduffel, and Yao (2014) and the references therein for more explanations. Moreover, closed-form approximations may also allow for efficient computation of the option Greeks and implied parameters, which are of great importance for hedging purposes as well as the pricing of new instruments. Finally, analytical approximations can also be useful in developing control variates to increase the efficiency of these Monte Carlo schemes; see Kemna and Vorst (1990), Fusai and Meucci (2008) and Dinger and Hörmann (2012).

Our contributions can be summarized as follows. *First*, based on conditioning techniques and Gaussian quadrature, we obtain new closed-form approximation formulas for pricing spread options in the Black-Scholes (BS), the Variance Gamma (VG) and the Normal Inverse Gaussian (NIG) set-up of the financial market. Numerical experiments show that our approximation is accurate and outperforms other known analytical approximations. In passing we also extend the approximations proposed by Carmona and Durrleman (2003a) and Bjerksund and Stensland (2011) in the context of a BS model to the VG and NIG models. The idea to deal with pricing of options using conditioning draws its inspiration from Rogers and Shi (1995) who derive in a BS model approximations for the value of an Asian option by conditioning on the average return. Similar ideas have also been used in the literature on portfolio optimization (Vanduffel, Dhaene, Goovaerts, & Kaas, 2003), annuity valuation (Vanduffel, Shang, Henrard, Dhaene, & Valdez, 2008) and pricing of basket options (Deelstra, Liinev, & Vanmaele, 2004). As for the idea of using Gaussian quadrature in option pricing, this technique finds its pedigree in Madan, Pistorius, and Schoutens (2011). *Second*, we extend Margrabe's (1978) seminal formula for the price of an exchange option (spread option with zero strike) from the BS setting to the VG setting under a certain homogeneity assumption.

The structure of the paper is as follows. In Section 2, we set out the spread option pricing problem and specify the market setting considered. The derivation of the closed-form approximation formulas, for which we rely on conditioning techniques and Gaussian quadrature rules, is presented in Section 3. In Section 4, the scope is reduced in that we focus solely on exchange options. After introducing the existing Margrabe formula valid in a BS model, its extension to a VG model is presented. Numerical results are discussed in Section 5. Section 6 concludes the paper.

2 Set-up

We consider a perfectly liquid and frictionless market with no arbitrage opportunities. There is a risk-free asset bearing a constant continuously compounded risk-free rate $r \geq 0$ and two risky assets with price processes $(S_i(t), t \geq 0)$ ($i = 1, 2$), defined on a probability space $(\Omega; \mathcal{F}; \{\mathcal{F}\}_{t \geq 0}; \mathbb{Q})$ ¹. Specifically, we model the price processes $(S_i(t), t \geq 0)$ as $S_i(t) = S_i \exp(X_i(t))$ where S_i denotes the asset price at $t = 0$ and the return processes $(X_i(t), t \geq 0)$ satisfy

$$X_i(t) = \mu_i t + \theta_i G(t) + \sigma_i Z_i(G(t)). \quad (1)$$

¹ $\{\mathcal{F}\}_{t \geq 0}$ is the filtration generated by $(S_i(t), t \geq 0)$ ($i = 1, 2$) and \mathbb{Q} is the pricing measure, used by market-participants for pricing derivatives that are written on underlying risky assets; see e.g., Cont and Tankov (2003).

Here, $\mu_i, \theta_i \in \mathbb{R}, \sigma_i \in \mathbb{R}^+$ and $(G(t), t \geq 0)$ is some non-negative increasing Lévy process that is independent of $(Z_1(t), Z_2(t))$, which is a two-dimensional standard Brownian motion with correlation coefficient ρ . Note that when $G(t) = t, t \geq 0$, we recover a two-dimensional BS model. The process $(G(t), t \geq 0)$ effectively time changes the underlying Brownian motion $(Z_1(t), Z_2(t))$ by randomizing the time stamp t and is also referred to as the subordinator; see e.g., Cont and Tankov (2003) and Luciano and Schoutens (2006). Sato (1999) shows that a Brownian motion (which is a particular Lévy process) that is time changed remains a Lévy process. Various choices are possible for the process $(G(t), t \geq 0)$, but in this paper we mainly concentrate on a Gamma process (Madan & Seneta, 1990) and an Inverse Gaussian process (Barndorff-Nielsen, 1995) as (non-degenerated) choices for $(G(t), t \geq 0)$. It is well known that time changed Brownian motions are suitable candidates to model log-returns of risky assets; see e.g., Prause (1999) and Schoutens (2003).

We consider a European call option on the price spread $S_1(T) - S_2(T)$ with strike $K \geq 0$ and maturity T . Its payoff at time T is given by

$$(S_1(T) - S_2(T) - K)^+$$

where $(\cdot)^+ = \max\{\cdot, 0\}$ and the price of the spread option at time $t = 0$ is then expressed as²

$$\begin{aligned} C(T, K) &= e^{-rT} E[(S_1(T) - S_2(T) - K)^+] \\ &= e^{-rT} E[(S_1 e^{X_1(T)} - S_2 e^{X_2(T)} - K)^+]. \end{aligned} \quad (2)$$

Note that (2) is a call-type payoff; however, the price of put-type spread options can easily be obtained by the call-put parity. The valuation of European-type spread options thus amounts to a two-dimensional integration problem with respect to the joint density function of the random asset return vector $(X_1(T), X_2(T))$. This is a non-trivial task and analytical solutions are not readily available.

Several approximation methods have been proposed in the literature, but most of them are only valid for the BS market. The most popular method, frequently used by practitioners, is the BS inspired approximation formula developed by Kirk (1995). Carmona and Durrleman (2003a) propose a pricing approach based on a set of lower bounds. They provide an upper bound as well and show numerically that the price range is very tight for certain parameter values. Li, Deng, and Zhou (2008) approximate the spread option price as a sum of one-dimensional integrals following the method introduced by Pearson (1995), in which the joint density is factored into the product of a univariate marginal and conditional density. Deelstra, Petkovic, and Vanmaele (2010) propose approximation formulas relying on comonotonicity theory and moment matching methods. Venkatramanan and Alexander (2011) approximate the spread option price by expressing it as the sum of the prices of two compound options. Finally, Bjerksund and Stensland (2011) show that the implicit strategy of the Kirk formula is to exercise when the long asset exceeds a scaled power function of the short asset, and use this insight to derive a closed-form pricing formula that yields a tight lower bound to the true spread option value. All of the above methods are valid under the assumption of asset log-returns that are normally distributed. In a non-Gaussian set-up, few explicit approaches exist. Apart from Monte Carlo simulations, Fast Fourier Transforms (FFT) can also be used to numerically evaluate spread options; see M. A. H. Dempster and Hong (2002), Hurd and Zhou (2010) and Caldana and Fusai (2013). In this regard, the pricing formulas that we present in Section 3 are alternatives to the use of Monte Carlo simulations or FFT-based techniques for pricing spread options in VG and NIG models.

²Note that we present our results in the context of pricing spread options, but technically our results hold with respect to any probability measure used for evaluating expectations, as appearing in display (2).

3 Closed-form approximations

In this section, we present closed-form approximation formulas for the pricing of spread options. First, we set out the general idea. Next, we develop approximation methods for BS, VG and NIG models, respectively.

3.1 Outline of the approach

Our approach for developing closed-form approximations relies on two basic tools, namely *conditioning techniques* and *Gaussian quadrature*. In the BS model, after conditioning on one of the asset prices, the (conditional) payoff resembles a plain vanilla call option, which can be valued using the seminal BS formula valid for pricing plain vanilla calls. The price of the spread option then writes as a one-dimensional integral, which can be effectively evaluated using Gauss-Hermite quadrature. In the VG and NIG models, we condition on the stochastic clock $G(T)$, which implies that one can apply (conditionally) the result already obtained for the BS model. Also in this case, the price of the spread option can be expressed as a one-dimensional integral, which can be effectively evaluated using Gauss-Laguerre quadrature.

A quadrature formula approximates the integral of a function $f(x)$ by the sum of its functional values at a set of n points, called nodes, multiplied by certain aptly chosen weighting coefficients, determined by a polynomial that interpolates the functional values. The nodes and weights are denoted as x_i and w_i ($i = 1, \dots, n$), respectively, and n is called the order of approximation. Specifically, a Gaussian quadrature rule for integrating $f(x)$ over $x \in [a, b]$ can be written as:

$$\int_a^b g(x)\omega(x)dx \approx \sum_{i=1}^n w_i g(x_i)$$

where $g(x)$ is (approximately) polynomial, the nodes x_i are the roots of the polynomial of order n associated with a known and positive weighting function $\omega(x)$, and the w_i are the corresponding weights. The Gaussian quadrature rules are constructed in such a way the quadrature formula is exact for all polynomials of degree $2n - 1$ or less by a suitable choice of the nodes x_i and weights w_i . It can be shown that, for the condition to hold, the x_i are the roots of a polynomial of degree n belonging to a specific class of orthogonal polynomials depending on the weighting function used, and the w_i are the associated weighting coefficients. In the next sections, we will use two specific Gaussian quadrature rules, namely Gauss-Hermite and generalized Gauss-Laguerre quadrature. To solve for the roots of these associated polynomials, the calculation of the corresponding weights and more details about these methods, we refer to Press, Teukolsky, Vetterling, and Flannery (1992).

3.2 The case of the BS model

The log-returns $(X_i(t), t \geq 0)$ ($i = 1, 2$) are as in equation (1) with $G(t) = t, t \geq 0$ and $\theta_1 = \theta_2 = 0$. In order to evaluate (2), we first condition on $X_2(T)$ and obtain

$$\begin{aligned} C(T, K) &= e^{-rT} E \left[E \left[(S_1 e^{X_1(T)} - (S_2 e^{X_2(T)} + K))^+ \mid X_2(T) \right] \right] \\ &= e^{-rT} \int_{-\infty}^{+\infty} E \left[(S_1 e^{X_1(T)} - (S_2 e^{x_2} + K))^+ \mid X_2(T) = x_2 \right] f_{X_2(T)}(x_2) dx_2 \\ &= e^{-rT} \int_{-\infty}^{+\infty} \left(\int_{k^*}^{+\infty} (S_1 e^{x_1} - (S_2 e^{x_2} + K)) f_{X_1(T) \mid X_2(T)=x_2}(x_1) dx_1 \right) f_{X_2(T)}(x_2) dx_2. \end{aligned} \quad (3)$$

Here, $k^* = \ln((S_2 e^{x_2} + K)/S_1)$, and the notation $f(\cdot)$ is used to depict in equation (3) the relevant densities. It is clear that both $X_1(T) \mid X_2(T) = x_2$ and $X_2(T)$ have normal densities with respective means $\mu_{1|2} = \mu_1 T + \frac{\rho \sigma_1}{\sigma_2} (x_2 - \mu_2 T)$ and $\mu_2 T$, and standard deviations $\sigma_{1|2} = \sqrt{\sigma_1^2 T (1 - \rho^2)}$ and $\sigma_2 \sqrt{T}$. After some rearrangements, we obtain

$$C(T, K) = e^{-rT} \int_{-\infty}^{+\infty} \left(S_1 e^{\frac{\sigma_{1|2}^2}{2} + \mu_{1|2} x_2} \Phi \left(\frac{\mu_{1|2} + \sigma_{1|2}^2 - k^*}{\sigma_{1|2}} \right) - (S_2 e^{x_2} + K) \Phi \left(\frac{\mu_{1|2} - k^*}{\sigma_{1|2}} \right) \right) f(x_2) dx_2 \quad (4)$$

with $\Phi(\cdot)$ the cumulative standard normal distribution function. By expressing the term between brackets in equation (4) as $\Pi(x_2)$, a function of x_2 , and implementing the change of variable $x_2 = \sqrt{2T} \sigma_2 \tilde{x}_2 + \mu_2 T$, we obtain

$$C(T, K) = \frac{e^{-rT}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \Pi(\sqrt{2T} \sigma_2 \tilde{x}_2 + \mu_2 T) e^{-\tilde{x}_2^2} d\tilde{x}_2. \quad (5)$$

An analytical expression for (5) is out of reach, but the integral is particularly suitable to be approximated numerically by the Gauss-Hermite quadrature rule. This rule is indeed characterized by the weighting function $\omega(x) = e^{-x^2}$ for $x \in [-\infty, +\infty]$. By applying the Gauss-Hermite rule to equation (5), we obtain our closed-form approximation formula for the price of a spread call option in BS setting:

$$C(T, K) \approx \frac{e^{-rT}}{\sqrt{\pi}} \sum_{i=1}^{n_1} w_i^{GH} \Pi(\sqrt{2T} \sigma_2 x_i^{GH} + \mu_2 T) \quad (6)$$

where n_1 is the chosen order of approximation (see also Section 5), $x_i^{GH} (i = 1, \dots, n_1)$ are the (known) zeros of the Hermite polynomial of order n_1 , $H_{n_1}(x)$, and $w_i^{GH} (i = 1, \dots, n_1)$ are the (known) corresponding weights (Abramowitz & Stegun, 1968).

3.3 The case of the VG model

The log-returns $(X_i(t), t \geq 0)$ ($i = 1, 2$) are as in equation (1) with $(G(t), t \geq 0)$ a Gamma process with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$. Specifically,

$$f_{G(t)}(g; \alpha t, \beta) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} g^{\alpha t - 1} e^{-g\beta}, \quad g > 0, \quad (7)$$

where $\Gamma(\cdot)$ stands for the gamma function. In order to evaluate (2), in a first step, by conditioning on the stochastic clock $G(T)$, we obtain

$$\begin{aligned} C(T, K) &= e^{-rT} E \left[E \left[(S_1 e^{X_1(T)} - (S_2 e^{X_2(T)} + K))^+ \mid G(T) \right] \right] \\ &= e^{-rT} \int_0^{+\infty} E \left[(S_1 e^{X_1(T)} - (S_2 e^{X_2(T)} + K))^+ \mid G(T) = g \right] f_{G(T)}(g) dg. \end{aligned} \quad (8)$$

Using that $(X_1(T), X_2(T) \mid G(T) = g)$ is bivariate normally distributed with means $\mu_i T + g\theta_i$ and standard deviations $\sqrt{g}\sigma_i$ ($i = 1, 2$), we obtain from equation (6) that

$$C(T, K) \approx e^{-rT} \int_0^{+\infty} \left(\frac{1}{\sqrt{\pi}} \sum_{i=1}^{n_1} w_i^{GH} \Pi(\sqrt{2g}\sigma_2 x_i^{GH} + \mu_2 T + g\theta_2) \right) f_{G(T)}(g) dg. \quad (9)$$

In a second step, we let g vary. Let us denote the expression within brackets in equation (9) as $\Upsilon(g)$, a function of g , i.e.,

$$\Upsilon(g) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n_1} w_i^{GH} \Pi(\sqrt{2g}\sigma_2 x_i^{GH} + \mu_2 T + g\theta_2),$$

then we obtain that

$$C(T, K) \approx e^{-rT} \int_0^{+\infty} \Upsilon(g) f_{G(T)}(g) dg. \quad (10)$$

By applying the density function of $G(T)$ (see equation (7)) and substituting $u = g\beta$, we find

$$C(T, K) \approx \frac{e^{-rT}}{\Gamma(\alpha T)} \int_0^{+\infty} \Upsilon\left(\frac{u}{\beta}\right) u^{\alpha T - 1} e^{-u} du. \quad (11)$$

This integral is particularly suitable to be computed numerically by the generalized Gauss-Laguerre quadrature rule, characterized by the weighting function $\omega(x) = x^k e^{-x}$ with $k > -1$ and $x \in [0, +\infty]$. Application of this rule to equation (11) yields the closed-form approximation formula:

$$C(T, K) \approx \frac{e^{-rT}}{\Gamma(\alpha T)} \sum_{j=1}^{n_2} w_j^{GL} \Upsilon(x_j^{GL})$$

where n_2 reflects the chosen order of approximation (see also Section 5), x_j^{GL} ($j = 1, \dots, n_2$) represent the (known) roots of the generalized Laguerre polynomial of order n_2 , $L_{n_2}^{(k)}(x)$, with $k = \alpha T - 1$, and w_j^{GL} ($j = 1, \dots, n_2$) are the (known) corresponding weights (Abramowitz & Stegun, 1968).

3.4 The case of the NIG model

The log-returns $(X_i(t), t \geq 0)$ ($i = 1, 2$) are as in equation (1) with $(G(t), t \geq 0)$ an Inverse Gaussian process with density function

$$f_{G(t)}(g; \delta t, \gamma) = \frac{\delta t}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(\gamma g - \delta t)^2}{g}} g^{-3/2}, \quad g > 0, \quad (12)$$

where $\gamma > 0$ and $\delta > 0$ (Barndorff-Nielsen, 1998). The pricing problem (2) can be approached in a similar manner as for the VG model to obtain equation (10). By applying the density function of $G(T)$ (see equation (12)) and substituting $u = \gamma^2 g/2$, we find

$$C(T, K) \approx \frac{\gamma \delta T e^{(\gamma \delta - r)T}}{2\sqrt{\pi}} \int_0^{+\infty} \Upsilon\left(\frac{2u}{\gamma^2}\right) u^{-3/2} e^{-\frac{(\gamma \delta T)^2}{4u}} e^{-u} du.$$

By denoting

$$\Psi(u) = \Upsilon\left(\frac{2u}{\gamma^2}\right) u^{-3/2} e^{-\frac{(\gamma \delta T)^2}{4u}},$$

we obtain that

$$C(T, K) \approx \frac{\gamma \delta T e^{(\gamma \delta - r)T}}{2\sqrt{\pi}} \int_0^{+\infty} \Psi(u) e^{-u} du. \quad (13)$$

This integral is particularly suitable to be computed numerically by the generalized Gauss-Laguerre quadrature rule (in which $k = 0$). Application of this rule to equation (13) yields the closed-form approximation formula:

$$C(T, K) \approx \frac{\gamma \delta T e^{(\gamma \delta - r)T}}{2\sqrt{\pi}} \sum_{j=1}^{n_2} w_j^{GL} \Psi(x_j^{GL})$$

where n_2 reflects the chosen order of approximation (see also Section 5), x_j^{GL} ($j = 1, \dots, n_2$) represent the (known) roots of the Laguerre polynomial of order n_2 , $L_{n_2}(x)$, and w_j^{GL} ($j = 1, \dots, n_2$) are the (known) corresponding weights (Abramowitz & Stegun, 1968).

4 A Margrabe-type formula for a VG market

Spread option contracts for which the strike price K equals to zero are called exchange options since they effectively give indeed the owner the right to exchange in future one asset for another. Margrabe (1978) points out that exchange options can be used in the valuation of the performance incentive fee, the general margin account, exchange offers and standby commitments. They are also building blocks for real options; see Vollert (2001).

4.1 Margrabe's formula in the BS model

In contrast to the general spread option pricing problem, there exists an analytical exact formula, i.e., Margrabe's formula, for the price of an exchange option in a BS set-up of the financial market. Margrabe's formula (1978) can be obtained (in a slightly extended form) from equation (4) by substituting the equations for $\mu_{1|2}$, $\sigma_{1|2}$ and k^* , and, after rearrangement of the expression found, applying the property

$$\int_{-\infty}^{+\infty} \Phi(a + bz) f(z; \mu, \sigma^2) dz = \Phi\left(\frac{a + b\mu}{\sqrt{1 + b^2\sigma^2}}\right)$$

where Φ and $f(z; \mu, \sigma^2)$ are the cumulative standard normal distribution function and the density function of a normal distribution with mean μ and variance σ^2 , respectively. The result is the

well-known Margrabe formula:

$$C(T, 0) = \sum_{i=1}^2 (-1)^{i+1} S_i e^{(\mu_i + \frac{\sigma_i^2}{2} - r)T} \Phi \left(\frac{\ln \left(\frac{S_1}{S_2} \right) + (\mu_1 - \mu_2)T + (-1)^{i+1} (\sigma_i^2 - \rho \sigma_1 \sigma_2)T}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2)T}} \right). \quad (14)$$

4.2 An extension of Margrabe's formula for the VG model

Recall that the two-dimensional VG model is described by price processes $S_i(t) = S_i \exp(X_i(t))$ where S_i is the asset price at $t = 0$ and in which the return processes $(X_i(t), t \geq 0)$ ($i = 1, 2$) satisfy

$$X_i(t) = \mu_i t + \theta_i G(t) + \sigma_i Z_i(G(t)).$$

Here, $\mu_i, \theta_i \in \mathbb{R}, \sigma_i \in \mathbb{R}^+$ and $(G(t), t \geq 0)$ is a Gamma process with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$ that is independent of $(Z_1(t), Z_2(t))$, which is a two-dimensional standard Brownian motion with correlation coefficient ρ . We define auxiliary parameters u_i and v_i ($i = 1, 2$) as follows:

$$u_i := \frac{\Delta\theta + (-1)^{i+1} \Delta\sigma_i}{\sqrt{\Delta\sigma_1 + \Delta\sigma_2}}$$

with $\Delta\mu := \mu_1 - \mu_2$, $\Delta\theta := \theta_1 - \theta_2$ and $\Delta\sigma_i := \sigma_i^2 - \rho \sigma_1 \sigma_2$, and

$$v_i := \frac{\sigma_i^2}{2} + \theta_i - \beta.$$

We make the following homogeneity assumption.

Assumption 1. *We assume that $S_1 = S_2$ and $\mu_1 = \mu_2$.*

The assumption that $S_1 = S_2$ may look restrictive but is practically relevant. For instance, when an asset manager may expect a specific fund in his portfolio to perform temporarily worse than a certain reference but does not want to sell this fund (e.g., because of transaction costs), he can obtain protection by the purchase of an exchange option that allows him to receive the positive difference between the value of the reference and his own fund. In this case, the nominal amounts of the underlying assets (portfolios) will be typically matched. Furthermore, the requirement $\mu_1 = \mu_2$ effectively means that our model will have one degree of freedom less than the full VG model. In fact, the original specification of the VG model in Madan, Carr, and Chang (1998) did not contain the additional deterministic terms (i.e., $\mu_1 = \mu_2 = 0$) and our model is thus somewhere in between the original model and the one described above. Note also that calibrating an eight-parameter model instead of a nine-parameter model to option data does not necessarily lead to a worse result.

Theorem 1 (Exact formula for the price of an exchange option in a VG model). *Under Assumption 1, the price of an exchange option in a VG economy is given as*

$$C(T, 0) = \sum_{i=1}^2 (-1)^{i+1} S_i e^{(\mu_i - r)T} \frac{\beta^{\alpha T}}{2(-v_i)^{\alpha T}} \left(1 + \frac{\sqrt{2} u_i \Gamma(\alpha T + 1/2) {}_2F_1 \left(\frac{1}{2}, \alpha T + \frac{1}{2}, \frac{3}{2}; \frac{u_i^2}{2v_i} \right)}{\sqrt{\pi(-v_i)} \Gamma(\alpha T)} \right). \quad (15)$$

Here, $\Gamma(\cdot)$ stands for the gamma function and ${}_2F_1$ is the so-called hypergeometric function (Abramowitz & Stegun, 1968, p.556).

Proof. Consider the pricing problem (2) with the strike price $K = 0$. By conditioning on the stochastic clock, $G(T)$, we obtain

$$\begin{aligned} C(T, 0) &= e^{-rT} E \left[E \left[(S_1 e^{X_1(T)} - S_2 e^{X_2(T)})^+ \mid G(T) \right] \right] \\ &= e^{-rT} \int_0^{+\infty} E \left[(S_1 e^{X_1(T)} - S_2 e^{X_2(T)})^+ \mid G(T) = g \right] f_{G(T)}(g) dg. \end{aligned} \quad (16)$$

Using that $(X_1(T), X_2(T) \mid G(T) = g)$ is bivariate normally distributed with means $\mu_i T + g\theta_i$, standard deviations $\sqrt{g}\sigma_i$ ($i = 1, 2$) and correlation coefficient ρ , we can apply the classical Margrabe formula (14) to equation (16) to obtain

$$C(T, 0) = \sum_{i=1}^2 (-1)^{i+1} S_i e^{(\mu_i - r)T} E \left[e^{\left(\frac{\sigma_i^2}{2} + \theta_i\right)G(T)} \Phi \left(\frac{\ln \left(\frac{S_1}{S_2}\right) + \Delta\mu T + \Delta\theta G(T) + (-1)^{i+1} \Delta\sigma_i G(T)}{\sqrt{(\Delta\sigma_1 + \Delta\sigma_2)G(T)}} \right) \right]. \quad (17)$$

We further denote the expectation in equation (17) as $E[f_i(G(T))]$ ($i = 1, 2$). Taking into account Assumption 1, we obtain that

$$\begin{aligned} E[f_i(G(T))] &= \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} \int_0^{+\infty} e^{(\frac{\sigma_i^2}{2} + \theta_i - \beta)g} g^{\alpha T - 1} \Phi \left(\frac{\Delta\theta + (-1)^{i+1} \Delta\sigma_i}{\sqrt{\Delta\sigma_1 + \Delta\sigma_2}} \sqrt{g} \right) dg \\ &= \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} \int_0^{+\infty} e^{v_i g} g^{\alpha T - 1} \Phi(u_i \sqrt{g}) dg. \end{aligned}$$

Decomposing the cumulative normal distribution function and expanding the exponential term by its Maclaurin series, i.e.,

$$\Phi[u_i \sqrt{g}] = \frac{1}{2} + \int_0^{u_i \sqrt{g}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j u_i^{2j+1} g^{j+1/2}}{2^j (2j+1) \Gamma(j+1)},$$

we obtain that

$$E[f_i(G(T))] = \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} \left(\frac{1}{2} \int_0^{+\infty} e^{v_i g} g^{\alpha T - 1} dg + \int_0^{+\infty} \frac{e^{v_i g} g^{\alpha T - 1}}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j u_i^{2j+1} g^{j+1/2}}{2^j (2j+1) \Gamma(j+1)} dg \right). \quad (18)$$

Note that the integrands on the right-hand side of (18) can be rewritten in terms of Gamma functions (involving some rescaling). We find that

$$E[f_i(G(T))] = \frac{\beta^{\alpha T}}{2(-v_i)^{\alpha T}} \left(1 + \frac{\sqrt{2} u_i}{\sqrt{\pi(-v_i)} \Gamma(\alpha T)} \sum_{j=0}^{\infty} \frac{(-1)^j u_i^{2j} \Gamma(j + \alpha T + 1/2)}{(-v_i)^j 2^j (2j+1) \Gamma(j+1)} \right).$$

Here we have also used that $v_1 < 0$ and $v_2 < 0$, since otherwise $E[f_i(G(T))]$ and thus also $C(T, 0)$ do not exist (see also expression (7) for the density of a Gamma distributed variable in which $g > 0$ must hold). Using the hypergeometric function we can further simplify the latter expression for $E[f_i(G(T))]$ ($i = 1, 2$) and after substitution in equation (17) we obtain the analytical Margrabe formula (15) for the VG model. \square

5 Numerical illustrations

In order to illustrate the methods discussed in this paper, we used the `smi.stocks` data included in the R-package **ghyp** (Luethi & Breymann, 2016). This dataset contains daily returns from January 2000 to January 2007 of the Swiss Market Index (SMI) and of five swiss blue chips, among which Swiss Reinsurance Company Ltd (Swiss Re) and Credit Suisse (CS). In the examples below, these stocks represent asset one and two, respectively. We parameterize the BS, VG and NIG models using a modification of the expectation maximization algorithm (A. P. Dempster, Laird, & Rubin, 1977), i.e., the Multi-Cycle Expectation Conditional Maximization algorithm has been used; for a detailed description we refer to McNeil, Frey, and Embrechts (2005). To obtain parameters under \mathbb{Q} , the Esscher transform (Gerber & Shiu, 1994) has been used; we refer to Section 3 in Deelstra et al. (2010) for more details. The calculations were performed on a PC with an Intel Core i7-4610M (3 GHz) processor and 16 GB of RAM.

5.1 BS model

Set-up: We compare the option prices obtained via our closed-form approximation formula with some benchmark prices obtained via the seminal Kirk formula (1995) and the pricing methods presented in Carmona and Durrleman (2003a) and Bjerk Sund and Stensland (2011). More details on these pricing methods can be found in Appendix A. Also, the price obtained via a Monte Carlo simulation with 10^6 trials is presented. Here, the standard error on the estimates obtained by simulation is reduced by using the lower bound developed by Bjerk Sund and Stensland (2011) as a control variate. Standard errors on the estimates obtained by simulation are provided as well. Our closed-form approximation formula relies on Gaussian quadrature and we also estimate the associated error that results from using quadrature formulas. A common approach in this regard is to calculate the difference of the results obtained by the Gaussian quadrature rule under consideration and a second higher-order quadrature rule; see e.g., Piessens, de Doncker-Kapenga, Überhuber, and Kahaner (1983). For this purpose, Gauss-Kronrod quadrature rules are often used. Gauss-Kronrod quadrature rules are higher-order extensions of Gaussian rules generated by adding $n + 1$ nodes to the existing n nodes constructed in such a way that the resulting rule has a degree of exactness of at least $3n + 1$. The extra nodes are the real zeros of the associated Stieltjes polynomial. This allows for computing higher-order estimates while reusing the function values of a lower-order estimate. Unfortunately, for the Gauss-Hermite rule, the associated Stieltjes polynomials have complex zeros for many values of n so that real positive Kronrod extensions do not readily exist. Therefore, to approximate the error, we calculate the difference between the Gauss-Hermite and the modified Kronrod-type extended Gauss-Hermite rule as developed by Begumisa and Robinson (1991). Their method relies on modification of the Stieltjes polynomial so that the resulting polynomial has no complex zeros and is proven to have only slightly lower degree of exactness than normally achieved when the Kronrod extension rule exists.

Prices are calculated for different combinations of strike K and exercise date T . We present prices for options having 20, 40, 60 and 120 days to maturity. Note that the order of approximation n_1 is the only quantity that one needs to specify in order to use the closed-form pricing formula. Based on experiments, it appears that in a BS setting as few as 2^4 nodes are sufficient, i.e., option prices thus converge very fast as the order of approximation is increased. Moreover, using $n_1 = 2^4$ makes it possible to use the values for the nodes and weights of the modified

Black-Scholes model							
		K					
		-20	-10	0	10	20	30
$T = 20$	CF	29.9875	20.0605	10.7197	3.7749	0.7120	0.0646
	C	0.0010	$< 5 \times 10^{-5}$	0.0010	0.0010	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	ER	1.42×10^{-14}	3.20×10^{-13}	2.66×10^{-11}	1.19×10^{-11}	3.01×10^{-12}	3.23×10^{-13}
	KI	29.9876	20.0608	10.7197	3.7750	0.7151	0.0656
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	BS	29.9875	20.0605	10.7197	3.7749	0.7118	0.0645
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	CD	29.9875	20.0605	10.7197	3.7749	0.7119	0.0646
	C	0.0010	0.0010	$< 5 \times 10^{-5}$	0.0010	$< 5 \times 10^{-5}$	0.0010
	MC	29.9875	20.0605	10.7197	3.7749	0.7120	0.0646
	C	0.8876	0.8986	0.8282	0.8976	0.9732	0.8816
	SE	1.14×10^{-6}	1.19×10^{-6}	3.79×10^{-8}	2.63×10^{-6}	7.04×10^{-6}	6.84×10^{-6}
$T = 40$	CF	30.0486	20.4212	11.7864	5.3392	1.7753	0.4225
	C	$< 5 \times 10^{-5}$	0.0010	0.0010	$< 5 \times 10^{-5}$	0.0010	0.0010
	ER	2.59×10^{-13}	1.54×10^{-12}	2.97×10^{-11}	1.58×10^{-11}	6.05×10^{-12}	1.52×10^{-12}
	KI	30.0491	20.4223	11.7864	5.3393	1.7809	0.4265
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	BS	30.0486	20.4211	11.7864	5.3390	1.7747	0.4218
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	CD	30.0486	20.4211	11.7864	5.3390	1.7748	0.4220
	C	0.0010	0.0010	$< 5 \times 10^{-5}$	0.0010	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	MC	30.0486	20.4212	11.7864	5.3391	1.7753	0.4225
	C	0.8966	0.9149	0.7976	1.0237	1.0857	0.8916
	SE	4.75×10^{-6}	3.87×10^{-6}	5.84×10^{-8}	6.36×10^{-6}	1.68×10^{-5}	2.17×10^{-5}
$T = 60$	CF	30.2300	20.9282	12.7480	6.5392	2.7367	0.9283
	C	$< 5 \times 10^{-5}$	0.0010	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	ER	7.67×10^{-13}	2.95×10^{-12}	3.19×10^{-11}	1.84×10^{-11}	8.32×10^{-12}	2.81×10^{-12}
	KI	30.2312	20.9302	12.7480	6.5395	2.7440	0.9351
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	BS	30.2298	20.9280	12.7480	6.5388	2.7354	0.9267
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	CD	30.2299	20.9280	12.7480	6.5388	2.7356	0.9271
	C	0.0010	0.0010	0.0010	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	MC	30.2301	20.9282	12.7480	6.5392	2.7367	0.9284
	C	0.8986	0.9126	0.7926	1.0787	0.9167	0.9257
	SE	1.24×10^{-5}	7.78×10^{-6}	2.45×10^{-7}	1.04×10^{-5}	2.77×10^{-5}	3.96×10^{-5}
$T = 120$	CF	31.1652	22.5829	15.1303	9.2459	5.1263	2.5928
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	0.0010	0.0010	$< 5 \times 10^{-5}$
	ER	2.86×10^{-12}	6.66×10^{-12}	3.62×10^{-11}	2.32×10^{-11}	1.28×10^{-11}	6.06×10^{-12}
	KI	31.1681	22.5868	15.1303	9.2467	5.1370	2.6047
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	BS	31.1638	22.5823	15.1303	9.2449	5.1226	2.5869
	C	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$
	CD	31.1640	22.5823	15.1303	9.2449	5.1229	2.5877
	C	0.0010	0.0010	$< 5 \times 10^{-5}$	$< 5 \times 10^{-5}$	0.0010	$< 5 \times 10^{-5}$
	MC	31.1652	22.5829	15.1303	9.2458	5.1263	2.5927
	C	0.8966	0.9197	0.7826	0.9482	0.9127	0.9927
	SE	4.05×10^{-5}	1.94×10^{-5}	2.24×10^{-8}	2.42×10^{-5}	6.51×10^{-5}	9.95×10^{-5}

Table 1: Summary of the results for the Black-Scholes model with $S_1 = 110$, $S_2 = 100$, $r = 0.01/252$, and parameters $\mu_1 = \mu_2 = -0.0002$, $\sigma_1 = 0.0211$, $\sigma_2 = 0.0235$ and $\rho = 0.5902$ for $t = 1$. Prices obtained from the closed-form approximation developed in Section 3 are referred to as CF. ER represents the approximated error of the CF price due to application of Gauss-Hermite quadrature. Kirk's (1995), Bjerksund and Stensland's (2011), and Carmona and Durrleman's (2003a) methods are referred to as KI, BS and CD, respectively. The notation MC and SE refer to the Monte Carlo simulated price and the associated standard error, respectively, and C denotes the computation time in seconds.

Kronrod extension of the Gauss-Hermite rule, as presented in Begumisa and Robinson (1991), when estimating the approximation error. Our numerical results for the BS model are reported in Table 1. We also present computation times (in seconds) for all calculated prices as an indication of complexity. However, note that in addition to computer specifications, running times also depend on the efficiency of the particular code that is used. Also note that the time to obtain the quadrature rule data (roots of polynomials and associated weights) is not included in the computation times of prices calculated via our closed-form approximation formula, as this data only depends on the specified order of approximation (which is assumed to be held constant).

Discussion: From Table 1, one can see that in BS setting the prices that result from our closed-form approximation (CF) and the ones obtained via Monte Carlo simulations (MC) are equal in almost all cases, which indicates that our formula produces very accurate prices in this setting. This can also be observed from the reported approximated errors (ER) included in our prices (CF). Also, we see that for $K = 0$ all presented pricing methods produce the same (exact) result. This can be explained by the fact that they all, with the exception of our formula, collapse into the analytical Margrabe formula (1978). The least accurate prices (compared to the benchmark prices, labelled as “MC”) for general values of K , are, as expected, the ones obtained by Kirk’s formula (KI). Based on the presented results, it seems that Kirk’s method tends to overprice the spread options. The results also show that prices obtained via the methods of Carmona and Durrleman (2003a) and Bjerksund and Stensland (2011) (labelled as “CD” and “BS”, respectively) are very accurate but tend to slightly underprice the spread options for large values of K and T . This observation is interesting since our closed-form approximation in BS setting can also be used for pricing spread options with longer time to maturities.

5.2 VG and NIG models

Set-up: We provide prices of spread options in a VG and NIG economy using the closed-form approximation formulas that we presented in Section 3. In addition, we also provide prices that come from generalizing the methods of Carmona and Durrleman (2003a) and Bjerksund and Stensland (2011) – initially developed for BS models – to VG and NIG models. These generalizations are obtained using the same ideas as used to develop our closed-form approximations: By first conditioning on the stochastic clock, we can conditionally apply the original methods of Carmona and Durrleman (2003a) and Bjerksund and Stensland (2011). Afterwards, we account for the stochasticity of the clock by applying Gaussian quadrature; see Appendix A.4 for more details. We also perform Monte Carlo simulations to compute “true” option prices (based on 10^8 trials). Finally, note that for VG and NIG models, the estimation of the error inherent in our approximations is prevented by the fact that we rely on both Gauss-Hermite and (generalized) Gauss-Laguerre quadrature for computing the option prices. The orders of approximation are chosen as $n_1 = 2^4$ and $n_2 = 2^7$ (under the VG and NIG models, prices converge more slowly). The option prices for the VG and NIG models can be found in Table 2 and Table 3, respectively. Also here, computation times (in seconds) are reported as an indication of complexity. In this regard, note that the time to obtain the generalized Gauss-Laguerre quadrature rule data for obtaining the option prices under the VG model using the closed-form approximation formula is now included in the computation times, as the quadrature rule data depends on both the parameterization of the pricing problem and the specified order of approximation.

Discussion: In both the VG and NIG settings, one can see from the prices reported in Table 2 and 3 that the prices that result from our closed-form approximations (CF) are very

Variance Gamma model							
		K					
		-20	-10	0	10	20	30
$T = 20$	CF	20.0527	10.6466	3.5762	0.5910	0.0456	0.0020
	C	0.4989	0.4083	0.3843	0.4133	0.3833	0.3983
	AN	-	-	3.5762	-	-	-
	BS	20.0527	10.6466	3.5762	0.5910	0.0455	0.0019
	C	0.3397	0.3398	0.3562	0.3563	0.3622	0.3442
	CD	20.0527	10.6466	3.5762	0.5910	0.0456	0.0020
	C	0.5040	0.4095	0.4063	0.4023	0.4063	0.3963
	MC	20.0528	10.6464	3.5760	0.5910	0.0455	0.0019
	C	2300.1	2426.3	2268.2	2176.7	2182.8	2177.4
	SE	7.26×10^{-5}	0.0002	0.0005	0.0002	5.44×10^{-5}	1.06×10^{-5}
$T = 40$	CF	20.3744	11.6389	5.0734	1.5324	0.3075	0.0426
	C	0.4623	0.3644	0.3913	0.3873	0.3843	0.3833
	AN	-	-	5.0734	-	-	-
	BS	20.3742	11.6388	5.0734	1.5322	0.3072	0.0423
	C	0.3240	0.4208	0.3533	0.3452	0.3793	0.3603
	CD	20.3743	11.6389	5.0734	1.5323	0.3074	0.0425
	C	0.4375	0.3898	0.3973	0.3953	0.4093	0.4013
	MC	20.3744	11.6387	5.0731	1.5323	0.3075	0.0425
	C	2293.3	2435.7	2193.5	2184.5	2177.6	2193.3
	SE	0.0002	0.0005	0.0007	0.0004	0.0002	6.15×10^{-5}
$T = 60$	CF	20.8411	12.5486	6.2187	2.4061	0.7102	0.1629
	C	0.3746	0.3884	0.3943	0.3843	0.3803	0.3763
	AN	-	-	6.2187	-	-	-
	BS	20.8407	12.5484	6.2187	2.4058	0.7094	0.1622
	C	0.4342	0.3376	0.3482	0.3523	0.3512	0.3502
	CD	20.8408	12.5484	6.2187	2.4059	0.7098	0.1626
	C	0.5032	0.4610	0.4073	0.3943	0.3923	0.4043
	MC	20.8410	12.5485	6.2183	2.4062	0.7101	0.1630
	C	2301.3	2460.5	2184.5	2185.5	2183.8	2214.6
	SE	0.0004	0.0006	0.0009	0.0006	0.0003	0.0001
$T = 120$	CF	22.4015	14.8197	8.7959	4.6072	2.1225	0.8703
	C	0.4411	0.3674	0.3793	0.3743	0.3723	0.3743
	AN	-	-	8.7959	-	-	-
	BS	22.4001	14.8191	8.7959	4.6064	2.1198	0.8665
	C	0.3216	0.2940	0.3402	0.3393	0.3452	0.3533
	CD	22.4002	14.8191	8.7959	4.6065	2.1206	0.8680
	C	0.3900	0.3938	0.3903	0.3873	0.3963	0.4183
	MC	22.4012	14.8194	8.7955	4.6071	2.1226	0.8702
	C	2304.4	2417.1	2192.8	2176.8	2181.9	2188.8
	SE	0.0008	0.0011	0.0013	0.0009	0.0006	0.0004

Table 2: Summary of the results for the VG model with $S_1 = S_2 = 100$, $r = 0.01/252$, and parameters $\mu_1 = \mu_2 = 0$, $\sigma_1 = 0.0193$, $\sigma_2 = 0.0225$, $\rho = 0.5426$, $\theta_1 = -0.0001$, $\theta_2 = -0.0002$ and $\alpha = \beta = 0.8973$ for $t = 1$. Prices obtained from the closed-form approximation developed in Section 3 are referred to as CF. AN refers to the price obtained via the analytical Margrabe-type formula as presented in Section 4. The prices obtained via our generalized approximations of Bjerksund and Stensland (2011) and Carmona and Durrleman (2003a) are denoted as BS and CD respectively. MC and SE refer to the Monte Carlo simulated price and the associated standard error, respectively, and C denotes the computation time in seconds.

Normal Inverse Gaussian model							
		K					
		-20	-10	0	10	20	30
$T = 20$	CF	29.9940	20.0918	10.7910	3.8817	0.8308	0.1166
	C	0.0410	0.0420	0.0760	0.0550	0.0460	0.0460
	BS	29.9939	20.0916	10.7861	3.8738	0.8264	0.1153
	C	0.0150	0.0050	0.0050	0.0080	0.0050	0.0050
	CD	29.9940	20.0918	10.7910	3.8817	0.8307	0.1165
	C	0.0911	0.1031	0.2331	0.2072	0.3522	0.2061
	MC	29.9940	20.0919	10.7901	3.8815	0.8307	0.1167
	C	2146.5	2136.5	2589.5	2397.2	2123.6	2127.8
	SE	2.88×10^{-5}	9.27×10^{-5}	0.0009	0.0006	0.0003	0.0001
$T = 40$	CF	30.0818	20.5031	11.9338	5.5295	1.9466	0.5257
	C	0.0480	0.0440	0.0480	0.0510	0.0550	0.0470
	BS	30.0815	20.5024	11.9244	5.5180	1.9381	0.5214
	C	0.0050	0.0080	0.0060	0.0200	0.0050	0.0050
	CD	30.0817	20.5030	11.9338	5.5294	1.9462	0.5252
	C	0.1031	0.0911	0.1561	0.1871	0.1981	0.1851
	MC	30.0818	20.5031	11.9333	5.5291	1.9468	0.5256
	C	2126.4	2145.9	2397.0	2418.6	2143.2	2377.0
	SE	0.0001	0.0003	0.0011	0.0008	0.0005	0.0003
$T = 60$	CF	30.3018	21.0677	12.9625	6.7888	2.9530	1.0702
	C	0.0460	0.0460	0.0500	0.1207	0.0520	0.0470
	BS	30.3010	21.0664	12.9498	6.7746	2.9412	1.0625
	C	0.0540	0.0070	0.0090	0.0120	0.0140	0.0050
	CD	30.3016	21.0676	12.9625	6.7886	2.9521	1.0690
	C	0.0871	0.0871	0.1791	0.1891	0.1931	0.1941
	MC	30.3018	21.0675	12.9615	6.7884	2.9529	1.0703
	C	2133.9	2141.6	2403.5	2391.1	2140.5	2299.3
	SE	0.0002	0.0004	0.0013	0.0010	0.0007	0.0004
$T = 120$	CF	31.3738	22.8802	15.4957	9.6218	5.4470	2.8237
	C	0.0410	0.0430	0.0630	0.0520	0.0440	0.0440
	BS	31.3713	22.8775	15.4760	9.6012	5.4271	2.8061
	C	0.0040	0.0100	0.0080	0.0040	0.0050	0.0060
	CD	31.3729	22.8798	15.4957	9.6210	5.4441	2.8191
	C	0.0981	0.0851	0.1461	0.1751	0.2132	0.1962
	MC	31.3736	22.8800	15.4946	9.6213	5.4466	2.8237
	C	2133.8	2122.4	2398.4	2393.6	2149.2	2291.9
	SE	0.0006	0.0009	0.0018	0.0014	0.0011	0.0008

Table 3: Summary of the results for the NIG model with $S_1 = 110$, $S_2 = 100$, $r = 0.01/252$, and parameters $\mu_1 = -0.0003$, $\mu_2 = 0.0009$, $\sigma_1 = 0.0200$, $\sigma_2 = 0.0234$, $\rho = 0.5333$, $\theta_1 = 0.0002$, $\theta_2 = -0.0012$, $\delta = 0.6349$ and $\gamma = 0.6331$ for $t = 1$. Prices obtained from the closed-form approximation developed in Section 3 are referred to as CF. The prices obtained via our generalized approximations of Bjerksund and Stensland (2011) and Carmona and Durrleman (2003a) are denoted as BS and CD respectively. MC and SE refer to the Monte Carlo simulated price and the associated standard error, respectively, and C denotes the computation time in seconds.

close to the reported “true” prices (MC). Prices obtained by generalizing the methods from Carmona and Durrleman (2003a) and Bjerksund and Stensland (2011) (labelled as “CD” and “BS”, respectively) are also acceptable, but the closed-form approximations are outperforming.

6 Conclusion

In this paper, closed-form spread option pricing formulas are presented for the BS, the VG, and the NIG model of the financial market. In order to compute prices of spread options via our approximations, the only quantity one needs to specify is the order of approximation to be used in the Gaussian quadrature rule. The presented pricing formulas provide (very) tight approximations for all models considered. For pricing in the BS setting, our method performs slightly better than existing (already quite accurate) pricing formulas. To the best of our knowledge, in the context of VG and NIG models, our pricing formulas for spread options have no known analytical competitor and can be regarded as alternatives to existing FFT-based techniques and Monte Carlo simulations. We also extend the seminal Margrabe formula for pricing exchange options in the BS model to the more general VG model under a homogeneity assumption.

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A Approximation methods in BS setting

A.1 Kirk’s formula

The Kirk formula (1995) is the most widely used analytical approximation formula for pricing spread options in BS setting. In this formula $S_2(T) + K$ is considered as a log-normal random variable with variance weighted using the relative proportions of S_2 and K . Then the problem becomes a simple exchange option which can be priced by the Margrabe formula. This results in the following approximation to the spread call

$$C(T, K) = S_1 \Phi(d_1) - (S_2 + K e^{-rT}) \Phi(d_2),$$

with

$$\begin{cases} d_1 = \frac{\ln(S_1) - \ln(S_2 + K e^{-rT})}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T}, \\ d_2 = d_1 - \sigma \sqrt{T}, \\ \sigma = \sqrt{\sigma_1^2 - 2 \frac{S_2}{S_2 + K e^{-rT}} \rho \sigma_1 \sigma_2 + \left(\frac{S_2}{S_2 + K e^{-rT}} \right)^2 \sigma_2^2}. \end{cases}$$

Note that the Margrabe formula used here is a special case of equation (14) where μ_i ($i = 1, 2$) in BS setting is plugged in ($\mu_i = r - \sigma_i^2/2$).

A.2 Carmona and Durrleman's method

Carmona and Durrleman (2003a) derive a pricing formula for spread options based on simple properties of the multivariate normal distribution and convexity inequalities. They obtain a lower bound that provides a very precise approximation to the actual price. The authors use two standard normal and independent state variables Z_1 and Z_2 along with trigonometric functions to model the correlated Brownian motions. More specifically, asset returns are modeled in the following way: $X_1 = \sqrt{1 - \rho^2}Z_1 + \rho Z_2$ and $X_2 = Z_2$ with $\cos \phi = \rho$ and $\phi \in [0, \pi]$. The strategy applied is to exercise when

$$Y_{\theta^*} = \sin \theta^* Z_1 - \cos \theta^* Z_2 \leq d^*,$$

where θ^* and d^* are found numerically by maximizing the option value w.r.t. this strategy. Their lower bound, following this method, is given by

$$C(T, K) = S_1 \Phi(d^* + \sigma_1 \sqrt{T} \cos(\theta^* + \phi)) - S_2 \Phi(d^* + \sigma_2 \sqrt{T} \cos \theta^*) - K e^{-rT} \Phi(d^*).$$

A.3 Bjerk Sund and Stensland's method

Bjerk Sund and Stensland (2011) use the implicit strategy of the Kirk formula to derive a formula for the spread call. They show that this strategy is to exercise when the long asset exceeds a scaled power function of the short asset. Using this insight, they derive a lower bound to the true spread option value:

$$\begin{aligned} C(T, K) &= e^{-rT} E \left[(S_1(T) - S_2(T) - K) I \left(S_1(T) \geq \frac{a(S_2(T))^b}{E[(S_2(T))^b]} \right) \right] \\ &= e^{-rT} (F_1 \Phi(d_1) - F_2 \Phi(d_2) - K \Phi(d_3)), \end{aligned}$$

where $F_i = S_i e^{rT}$ ($i = 1, 2$) is the current forward price for delivery at the future date T , and d_1, d_2 and d_3 are defined by

$$\begin{cases} d_1 = \frac{\ln(F_1/a) + (\frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2)T}{\sigma\sqrt{T}}, \\ d_2 = \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2)T}{\sigma\sqrt{T}}, \\ d_3 = \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2)T}{\sigma\sqrt{T}}, \\ \sigma = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2}, \end{cases}$$

and where the constants a and b are given as

$$a = F_2 + K \text{ and } b = \frac{F_2}{F_2 + K}.$$

The authors also point out it is possible to optimize the values of a and b by maximizing the spread call value. Moreover, they prove that optimizing their lower bound with respect to a and b is essentially equivalent to the Carmona and Durrleman (2003a) method.

A.4 Generalization to VG and NIG models

The approximations of Carmona and Durrleman (2003a) and Bjerk Sund and Stensland (2011) can be generalized from a BS to a VG and NIG setting using the same ideas as used to develop our closed-form approximation formulas for these settings; see Sections 3.3 and 3.4, respectively.

More specifically, in order to evaluate the pricing problem (2) in a VG or NIG setting, we first condition on the stochastic clock $G(T)$ in order to obtain equation (8). Using that $(X_1(T), X_2(T) \mid G(T) = g)$ is bivariate normally distributed with means $\mu_i T + g\theta_i$, standard deviations $\sqrt{g}\sigma_i$ ($i = 1, 2$) and correlation coefficient ρ , we can apply conditionally either the original method of Carmona and Durrleman (2003a) or Bjerksund and Stensland (2011) in order to obtain equation (10). In this case, $\Upsilon(g)$, a function of g , is the (undiscounted) option price obtained via the approximation method chosen, instead of our approximation formula for the BS setting (see equation (6)). In a second step, we account for the stochasticity of the clock by applying Gaussian quadrature as set out in Sections 3.3 and 3.4 for the VG and NIG setting, respectively.